The Kuznetsov formula for GSp_4

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Introduction

The GL_2 Kuznetsov formula relates, for **fixed** integers $m, n \neq 0$ and h a "nice" test function, a sum of terms of the form

$$h(t_u)a_m(u)\overline{a_n(u)},$$

where u varies among **Hecke Maaß forms**, $a_m(u)$ is the m-th **Fourier coefficient** of u, and t_u is the spectral parameter of u to a sum of Kloosterman sums, plus a continuous contribution from Eisenstein series.

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- Maaß forms $\leftrightsquigarrow$ K_∞-fixed functions in cuspidal automorphic representations of GSp_4,
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Methods of proof:
- Inner product of Poincaré series: Kuznetsov (GL_2), Blomer, Buttcane (GL_3), Man (GSp_4),...
- Relative trace formula: Zagier (unpublished), Knightly-Li.

Outline

1. Automorphic forms on GSp_4
2. The trace formula
3. Applications

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The group GSp_4

$\mathrm{GSp}_4 = \{g \in \mathrm{GL}_4 : \exists \mu(g) \in \mathrm{GL}_1, {}^\top g J g = \mu(g) J\}$, where $J = \begin{bmatrix} & I_2 \\ -I_2 & \end{bmatrix}$.

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Parabolic subgroups $P = N_P M_P$:

- Borel subgroup B: $N_B = U = \begin{bmatrix} 1 & & * & * \\ * & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{bmatrix} \cap \mathrm{GSp}_4$,

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$K = K_\infty \prod_p K_p \subset \mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$, $K(N) = K_\infty \prod_p K_p \subset \mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$.

The Langlands spectral decomposition

Consider the representation of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ on $L^2(\mathbb{R}_{>0}\mathrm{GSp}_4(\mathbb{Q})\backslash\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q}))$ given by $g \cdot \phi = \phi(\cdot g)$. It decomposes as

$$L^2(\mathbb{R}_{>0}\mathrm{GSp}_4(\mathbb{Q})\backslash\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})) = \bigoplus_\omega L^2(\omega),$$

where ω runs over characters of $\mathbb{R}_{>0}\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times$ and $L^2(\omega)$ is subspace consisting in function ϕ that satisfy $\phi(gz) = \omega(z)\phi(g)$ for all $z \in \mathbb{A}_\mathbb{Q}^\times$.

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- L^2_{cont} is a direct integral of representations induced from parabolic subgroups by Eisenstein series attached to characters and to automorphic forms on $\mathrm{GL}_1 \times \mathrm{GL}_2$, respectively.
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An irreducible automorphic representation π is called **cuspidal** if for every parabolic P every $\phi \in \pi$ satisfies $\int_{N_P(\mathbb{Q})\backslash N_P(\mathbb{A})} \phi(ux)du = 0$ for all x.
Analogue of Maaß forms: K_∞-fixed elements of cuspidal representations π.

Whittaker coefficients

Let ψ be a fixed (generic) character of U. The ψ-Whittaker coefficient of ϕ is by definition

$$W_\phi(x) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A}_\mathbb{Q})} \phi(ux)\overline{\psi(u)}du.$$

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Unlike the case of GL_2, W_ϕ is not always non-zero, even if ϕ is not constant. For instance, Whittaker coefficients of Siegel modular forms are always zero.

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This is equivalent to say π has a ψ-**Whittaker model**, i.e, can be realized by right translation on a space of functions W with moderate growth and satisfying

$$W(ug) = \psi(u)W(g)$$

for all $u \in U$.

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The automorphic kernel

Let $f : \mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}}) \to \mathbb{C}$ be a smooth function satisfying $f(gz) = \overline{\omega(z)} f(g)$, compactly supported mod centre. Then we have an operator $R(f)$ on $L^2(\omega)$ defined by

$$(R(f)\phi)(x) = \int_{\overline{G}(\mathbb{A})} f(y)\phi(xy)dy = \int_{\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})} K_f(x,y)\phi(y)dy,$$

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Informally, we have $(R(f)\phi)(x) = \langle K_f(x,\cdot), \overline{\phi}\rangle$. So if $\mathscr{B}$ is an orthonormal basis of L^2_{disc} we expect

$$\begin{aligned} K_f(x,y) &= \sum_{\phi\in\mathscr{B}} \langle K_f(x,\cdot), \overline{\phi}\rangle \overline{\phi}(y) + cont. \\ &= \sum_{\phi\in\mathscr{B}} (R(f)\phi)(x)\overline{\phi}(y) + cont. \end{aligned}$$

The choice of the test function

Each irreducible automorphic representation π factors as $\pi \cong \bigotimes_{p \leq \infty} \pi_v$. If f also factors, then the operator $R(f)$ induces an operator $\pi_v(f_v)$ on the space of each representation π_v.

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For this choice of f, taking $\mathscr{B}_\pi(N)$ the basis of $\pi^{K(N)}$ obtained by the tensor product of the local basis, we obtain

$$K_f(x,y) = \sum_\pi \sum_{\phi\in\mathscr{B}_\pi(N)} \lambda_f(\phi)\phi(x)\overline{\phi}(y) + cont.$$

The spectral side

Integrating previous expression against $\overline{\psi}(x)\psi(y)$ over $(U(\mathbb{Q})\backslash U(\mathbb{A}_{\mathbb{Q}}))^2$ we obtain

$$\int_{(U(\mathbb{Q})\backslash U(\mathbb{A}_{\mathbb{Q}}))^2} K_f(xt_1, yt_2)\overline{\psi(x)}\psi(y)dxdy = \sum_{\pi} \sum_{\phi\in\mathscr{B}_\pi(N)} \lambda_f(\phi)W_\phi(t_1)\overline{W_\phi}(t_2) + cont.$$

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On the RHS, only representations π which are generic and have a non-zero $K(N)$-fixed vector contribute. It is known that the Archimedean component must then be a principal series representation, i.e, induced from a character $a \mapsto a^{\rho+\nu}$ of A.

The spherical transform

In the standard induced model, the K_∞-fixed vector ϕ_∞ is given by $\phi_\infty(nak) = a^{\rho+\nu}$, and hence the eigenvalue λ_{f,π_∞} is given by

$$\lambda_{f,\pi_\infty} = (\pi_\infty(f_\infty)\phi_\infty)(1) = \int_{\overline{G}(\mathbb{R})} f_\infty(g)\pi_\infty(g)\phi(1)dg$$
$$= \int_{U(\mathbb{R})}\int_{\overline{A}(\mathbb{R})} f(na)a^{\rho+\nu}dadn \doteq \tilde{f}(\nu),$$

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where $\tilde{f}$ is the **spherical transform** of f_∞. So the spectral side becomes

$$\sum_{\pi} \tilde{f}(\nu_\pi) \sum_{\phi\in\mathscr{B}_\pi(N)} \lambda_{f_{fin}}(\phi)W_\phi(t_1)\overline{W_\phi}(t_2) + cont.$$

where $\nu_\pi \in Lie(\overline{A})^*$ is the **spectral parameter** of π_∞.

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By Harish-Chandra Paley-Wiener's theorem, it is known that, choosing appropriately the test function f_∞, we can produce any Paley-Wiener test function $h = \tilde{f}$ on the spectral side.

The geometric side

From now on, we assume $t_1, t_2 \in A(\mathbb{Q})$. By definition of the kernel, the integral we considered may be written as

$$\sum_{\gamma\in\overline{G}(\mathbb{Q})} \int_{(U(\mathbb{Q})\backslash U(\mathbb{A}_\mathbb{Q}))^2} f(t_1^{-1}x^{-1}\gamma y t_2)\overline{\psi(x)}\psi(y)dxdy = \sum_{\gamma\in U(\mathbb{Q})\backslash\overline{G}(\mathbb{Q})/U(\mathbb{Q})} I_\gamma(f),$$

where

$$I_\gamma(f) = \int_{H_\gamma(\mathbb{Q})\backslash U(\mathbb{A}_\mathbb{Q})^2} f(x^{-1}t_1^{-1}\gamma t_2 y)\overline{\psi(t_1 x t_1^{-1})}\psi(t_2 y t_2^{-1})dxdy,$$

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$H_\gamma = \{(x, y) \in U^2, x^{-1}\gamma y = \gamma\}$. Using the Bruhat decomposition, $U(\mathbb{Q})\backslash \overline{G}(\mathbb{Q})/U(\mathbb{Q})$ consists in elements $\sigma\delta$, where σ ranges over the Weyl group, and δ over $A(\mathbb{Q})$.

Non-Archimedean part of the geometric side

By the bi-$K(N)$-invariance property of f, the non-Archimedean part of $I_{\sigma\delta}$ reduces to a finite sum $\mathrm{Kloos}_\sigma(\delta, f_{fin}, N)$. For simplicity, assume now that f_{fin} is "trivial".

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So the geometric side becomes

$$
\begin{aligned}
I_{1,\infty}(f_\infty)\delta(t_1, t_2) &+ \sum_{\delta} I_{\sigma_1\delta,\infty}(f_\infty)\mathrm{Kloos}_{\sigma_1}(\delta, N) \\
&+ \sum_{\delta} I_{\sigma_2\delta,\infty}(f_\infty)\mathrm{Kloos}_{\sigma_2}(\delta, N) \\
&+ \sum_{\delta} I_{\sigma_l\delta,\infty}(f_\infty)\mathrm{Kloos}_{\sigma_l}(\delta, N)
\end{aligned}
$$

Archimedean part of the geometric side

After a change of variable, the Archimedean part of $I_{\sigma\delta}(f)$ is given by

$$\int_{U_\sigma(\mathbb{R})\backslash U(\mathbb{R})} \int_{U(\mathbb{R})} f(u_1^{-1} t_1^{-1} \sigma\delta t_2 u_2) \overline{\psi(t_1 u_1^{-1} t_1^{-1})} \psi(t_2 u_2 t_2^{-1}) du_1 du_2.$$

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$$\int_{U_\sigma(\mathbb{R})\backslash U(\mathbb{R})}\int_{U(\mathbb{R})} f(u_1^{-1}t_1^{-1}\sigma\delta t_2 u_2)\overline{\psi(t_1 u_1^{-1} t_1^{-1})}\psi(t_2 u_2 t_2^{-1}) du_1 du_2.$$

Using Wallach's Whittaker inversion, we can show

$$\int_{U(\mathbb{R})} f(tug)\overline{\psi}(u)du = \int_{Lie(\overline{A})^*} \tilde{f}(\nu)W(-\nu, g, \psi)W(\nu, t^{-1}, \overline{\psi})dspec(\nu),$$

where $W(-\nu, \cdot, \psi)$ is the ψ-Whittakher function with spectral parameter $-\nu$ = the K_∞-fixed vector in the Whittakher model of the principal series representation with spectral parameter $-\nu$.

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Under some conjectural interchange of integral, the whole integral would become

$$\int_{Lie(\overline{A})^*}\tilde{f}(\nu)W(-\nu,t_2,\psi)W(\nu,t_1,\overline{\psi})K_\sigma(-i\nu,\delta)dspec(\nu),$$

where K_σ is a generalised Bessel function.

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2 The trace formula

3 Applications

Satake parameters

Fix a prime p and let π_p be an irreducible admissible representation of $\mathrm{GSp}_4(\mathbb{Q}_p)$ which has a K_p-fixed vector and trivial central character.
It is known that π_p is the unique spherical subquotient of a representation induced from a character of the form $\begin{bmatrix} x & & & \\ & y & & \\ & & tx^{-1} & \\ & & & ty^{-1} \end{bmatrix} \mapsto \sigma(t)\chi_1(x)\chi_2(y)$ for some unramified characters χ_1, χ_2, σ satisfying $\sigma^2\chi_1\chi_2 = 1$.

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π_p is **tempered** if and only if it Satake parameters $(\alpha, \beta) \in \mathbb{S}^1 \times \mathbb{S}^1$. It is believed that if π_p is the local constituent of the automorphic representation attached to a GSp_4 Maaß form, then π_p is tempered (Generalised Ramanujan Conjecture).

Equidistribution of Satake parameters

Let $\mathcal{F}(N) = \bigcup_\pi \mathscr{B}_\pi(N)$, orthonormal basis of the space of $K(N)$-fixed Maaß forms on $\mathrm{GSp}(\mathbb{A})$.

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$$\{(\alpha_{p,\phi}, \beta_{p,\phi}) : \phi \in \mathcal{F}(N)\} \subset \mathbb{C}^2/W,$$

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weighted by $w(\phi)$, equidistributes with respect to the GSp_4 Sato-Tate measure μ_{ST} as N tends to infinity among integers coprimes to p. This means that for any continuous bounded W-invariant function f on $\mathbb{C}^2$ we have

$$\lim_{N\to\infty} \frac{\sum_{\phi\in\mathcal{F}(N)} w(\phi) f(\alpha_{p,\phi}, \beta_{p,\phi})}{\sum_{\phi\in\mathcal{F}(N)} w(\phi)} = \int_{\mathbb{C}^2/W} f d\mu_{ST}.$$

This is consistent with the Generalised Ramanujan Conjecture.

Strategy

Taking $t_1 = 1$ and $t_2 = \begin{bmatrix} p^i & & & \\ & p^j & & \\ & & p^{k-i} & \\ & & & p^{k-j} \end{bmatrix}$, the cuspidal term in the spectral side of the Kuznetsov formula gives

$$\sum_{\phi \in \mathcal{F}(N)} w(\phi) f_{i,j,k}(\alpha_{p,\phi}, \beta_{p,\phi})$$

with $f_{i,j,k}(\alpha, \beta) = W_{\alpha,\beta}\left(\begin{bmatrix} p^i & & & \\ & p^j & & \\ & & p^{k-i} & \\ & & & p^{k-j} \end{bmatrix}\right)$, where $W_{\alpha,\beta}$ is the (normalized) Whittaker function for the representation of $\mathrm{GSp}_4(\mathbb{Q}_p)$ with Satake parameters (α, β).

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The continuous contributions

Formally similar to the cuspidal contribution with following modifications

$$\sum_{\pi\subset L^2_{disc}(\mathrm{GSp}_4)} \rightsquigarrow \int_{\nu\in i\mathrm{Lie}(\overline{A_P})^*} \sum_{\pi\subset L^2_{disc}(M_P)}$$

$$\phi\in\pi \rightsquigarrow E(\cdot,u,\nu)\in \mathrm{Ind}_P^{\mathrm{GSp}_4}\left(1_{N_P}\otimes\exp(\nu)\otimes\pi\right),$$

where u ranges over an ON basis $\mathscr{B}_\pi$ of the space $\mathscr{H}_P(\pi)$ of functions st

- u is left-invariant by $N_P(\mathbb{A})$,
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In particular, if $\gamma k \cdot K(N) = k \cdot K(N)$ then $\pi(\gamma)u_k = u_k$. Hence

$$\mathscr{H}_P(\pi) \simeq \bigoplus_{k\in(P(\mathbb{A})\cap K)\backslash K/K(N)} V_P(k,\pi)$$

$$u \mapsto (u_k)$$

where $V_P(k,\pi)$ is the (finite dimensional) space of vectors in π that are invariant by $\Gamma_{P,k}(N) \doteq Stab_{P(\mathbb{A})\cap K}(k \cdot K(N))$.

Orthonormal basis of $\mathscr{H}_P(\pi)$

The relevant inner product on $\mathscr{H}_P(\pi)$ is given by

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An optimization problem

We want to bound

$$\sum_{\pi \subset L^2_{disc}(M_P)} h(\nu + \nu_\pi) \sum_{k,i} W_{E(\cdot, u^{(k,i)}, \nu)}(t_1) \overline{W_{E(\cdot, u^{(k,i)}, \nu)}(t_2)}.$$

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and $\|u\|_\infty = \max_h \sup_m |u(mh)| = \max_h \|u_h\|_\infty$. Suppose we know $\|u_{h,j}\|_\infty \ll X$. We want to bound

$$\sum_{k,i} \left(\max_h \frac{1}{\sqrt{\#\mathcal{O}_h}} \sum_j |c_{h,j}^{(k,i)}| \|u_{h,j}\|_\infty \right)^2 \ll X^2 \sum_{k,i} \left(\max_h \frac{1}{\sqrt{\#\mathcal{O}_h}} \sum_j |c_{h,j}^{(k,i)}| \right)$$

The choice of an orthonormal basis

Take

$$u_h^{(k,i)} = \begin{cases} c_h u_{h,i} \text{ if } \#\mathcal{O}_k \approx \#\mathcal{O}_h \\ 0 \text{ otherwise,} \end{cases}$$

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we can take $|c_h|$ as small as $\frac{1}{\sqrt{d_h}}$ where $d_h = \#\{k : \#\mathcal{O}_k \approx \mathcal{O}_h\}$.

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$$u_h^{(k,i)} = \begin{cases} c_h u_{h,i} \text{ if } \#\mathcal{O}_k \approx \#\mathcal{O}_h \\ 0 \text{ otherwise,} \end{cases}$$

so that

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we can take $|c_h|$ as small as $\frac{1}{\sqrt{d_h}}$ where $d_h = \#\{k : \#\mathcal{O}_k \approx \mathcal{O}_h\}$.
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If $\#\mathcal{O}_k \approx \#\mathcal{O}_h$ then $d_h = d_k$, and hence the contribution from π is bounded by

$$X^2 \sum_{k,i} \frac{1}{d_k \#\mathcal{O}_k} \ll X^2 \sum_k \frac{\dim(V_P(k,\pi))}{d_k \#\mathcal{O}_k}.$$

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So the contribution from P is bounded by

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In reality, the argument is more complicated as there are small orbits.
Instead of bounding $\|u_{k,j}\|_\infty$ uniformly, we need a bound that depends on k.

Bounding the sums of Kloosterman sums

Because f has compact support, the set of δ's such that

$$\int_{U_\sigma(\mathbb{R})\backslash U(\mathbb{R})} \int_{U(\mathbb{R})} f(u_1^{-1}t_1^{-1}\sigma\delta t_2 u_2)\overline{\psi(t_1 u_1^{-1} t_1^{-1})}\psi(t_2 u_2 t_2^{-1}) du_1 du_2 \neq 0$$

is compact.

But the summation over δ is subject to some divisibility-by-N conditions.The upshot is as N gets large, only the identity contribution will remain on the geometric side (our formula is arguably more of a "pre-Kuznetsov" formula).

THANK YOU FOR YOUR ATTENTION!